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Interscale transfer in two-dimensional compact vortices

By GIANNI PEDRIZZETTI[1] AND J. C. VASSILICOS[2]

[1] Dipartimento Ingegneria Civile, Università di Trieste, P.le Europa 1, 34127 Trieste, Italy
e-mail: gianni@dic.univ.trieste.it

[2] Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK
e-mail: jcv10@damtp.cam.ac.uk

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The property of transfer between different scales of motion in evolving two-dimensional compact vortices is studied here, and a general mathematical framework is developed to describe the transfer between scales inside compact structures. This new approach is applied to the case of an axisymmetric advection which represents the leading-order (large time) approximation for Lundgren's family of two-dimensional vortices. It is also generalized to passive scalar advection by non-axisymmetric velocity fields. It is shown that scale interactions generated by an axisymmetric advection are essentially local and dominated by distant triadic interactions: in the case of an evolving spiral vortex sheet this result is confirmed even when non-axisymmetric corrections are included. A physical interpretation of the results is given, which can be summarized by saying that locality of scale interactions is caused by the uniformity of shear at a given scale and is therefore increasingly natural at small lengthscales. Local interactions are shown to arise in axisymmetric advection but to be uncommon in non-axisymmetric advection.

1. Introduction

Both in two-dimensional and three-dimensional turbulence as well as in geophysical flows there is a tendency for vorticity to organize itself into coherent compact structures. There is also evidence from direct numerical simulations of turbulence that both vortex tubes and sheets exist in the small scales of the turbulence and even that at least some vortex tubes may result from vortex sheet instabilities or self-induced dynamics (Vincent & Meneguzzi 1994; Passot *et al.* 1995). More generally, vortex spirals appear naturally because of vortex-sheet instabilities of self-induced roll-up and also spirals of passive scalars and of weak vorticity are continuously formed by filamentation or simply advection by a local differential rotation (see the recent paper by Gibbon, Fokas & Doering 1999).

It is in the spirit of modelling and studying compact vortex structures and their possible implications for turbulent flows and their statistics and scaling that Lundgren (1982) derived a general two-dimensional solution of the system of equations

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{u}\cdot\nabla\zeta = 0,$$
$$\zeta = \nabla \times \boldsymbol{u},$$
$$\nabla\cdot\boldsymbol{u} = 0,$$

where ζ is the vorticity normal to the plane of the two-dimensional incompressible velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$. We do not consider viscous effects in this paper, but such effects can be built into the solution and have in fact been the object of quite detailed study by Lundgren (1982), Flohr & Vassilicos (1997) and Angilella & Vassilicos (1999) (see also Pullin & Saffman 1998 and references therein).

In a two-dimensional polar coordinate system where the position $\boldsymbol{x}$ has radial and azimuthal coordinates r and θ and where u_r and u_θ are the radial and azimuthal components of the velocity $\boldsymbol{u}$, the velocity field $u_r = 0$, $u_\theta = r\,\Omega(r)$ is an exact solution of the above system of equations for any arbitrary suitably differentiable frequency $\Omega(r)$ of differential rotation. The vorticity of this velocity field is $\zeta = (1/r)(\mathrm{d}/\mathrm{d}r)(r^2\Omega(r))$. Lundgren's (1982) family of solutions is constructed by adding a time-dependent velocity field to this axisymmetric time-independent solution, that is

$$u_\theta = r\,\Omega(r) + \widetilde{u_\theta}(r, \theta, t),$$

$$u_r = 0 + \widetilde{u_r}(r, \theta, t),$$

and the vorticity becomes

$$\zeta(r, \theta, t) = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\Omega(r)) + \omega(r, \theta, t),$$

where

$$\omega(r, \theta, t) = \frac{1}{r}\frac{\partial}{\partial r}(r\widetilde{u_\theta}) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

is the correction vorticity. The incompressibility requirement implies the existence of a streamfunction $\psi(r, \theta, t)$ such that

$$\widetilde{u_\theta}(r, \theta, t) = -\frac{\partial \psi}{\partial r}, \tag{1a}$$

$$u_r(r, \theta, t) = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \tag{1b}$$

$$\omega = -\nabla^2\psi. \tag{1c}$$

The two-dimensional vorticity equation can be recast to read

$$\frac{\partial \omega}{\partial t} + \Omega(r)\frac{\partial \omega}{\partial \theta} + \frac{\widetilde{u_\theta}}{r}\frac{\partial \omega}{\partial \theta} + u_r\frac{\partial \omega}{\partial r} + u_r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\Omega(r))\right] = 0, \tag{2}$$

and because $\omega(r, \theta - \Omega(r)\,t)$ is the general solution of $\partial\omega/\partial t + \Omega(r)\,\partial\omega/\partial\theta = 0$, Lundgren (1982) sought a solution of (1)–(2) of the form $\omega(r, \theta - \Omega(r)\,t, t)$. The extra explicit time-dependence is crucial to Lundgren's family of solutions and Lundgren (1982) limited this family to cases where this explicit time-dependence can be expressed as a series of non-negative powers of t^{-1}, thereby obtaining solutions of (1)–(2) that are valid asymptotically in the limit $t \to \infty$. Lundgren (1982) added the requirement that the explicit time-dependence of $\omega(r, \theta - \Omega(r)\,t, t)$ should be $O(1)$ so that the vorticity field $\omega(r, \theta - \Omega(r)\,t, t)$ does not blow up nor tend to 0 as $t \to \infty$. It then follows from (1c) that the explicit time-dependence of $\psi(r, \theta - \Omega(r)\,t, t)$ is $O(t^{-2})$, that of u_r is $O(t^{-2})$, and that of u_θ is $O(t^{-1})$. Given the spiral wind-up form $\omega(r, \theta - \Omega(r)\,t, t)$, $\partial\omega/\partial r$ is $O(t)$ and $\partial\omega/\partial\theta$ is $O(1)$. Hence, the Lundgren family of solutions is such that the first two terms $\partial\omega/\partial t + \Omega(r)\partial\omega/\partial\theta$ on the left-hand side of (2) are $O(1)$, the third and fourth terms are $O(t^{-1})$ and the last term is $O(t^{-2})$.

Lundgren's family of solutions has been extended to three dimensions by applying an irrotational strain field in the direction normal to their two-dimensional plane (Lundgren 1982). Energy wavenumber spectra and higher-order moments of single and of particular ensembles of Lundgren vortices have been calculated and compared to their turbulence counterparts (Lundgren 1982; Gilbert 1988; Segel 1995; Saffman & Pullin 1996). Eulerian and Lagrangian frequency spectra of Lundgren vortices have also been calculated numerically with the result that the Eulerian frequency spectrum can be derived from the wavenumber spectrum using the Tennekes advection relation for turbulent flow (Malik & Vassilicos 1996). Attempts have been made to construct small-scale turbulence models from ensembles of specific Lundgren vortices, on the basis of which, statistics of one-point velocity derivatives and vorticity have been calculated and compared with numerical and laboratory experiments (see Pullin & Saffman 1998 and references therein). However, interscale energy transfer in Lundgren compact vortices is an issue that has not been addressed in the literature, even though interscale energy transfer is central to two-dimensional and three-dimensional turbulence dynamics (see for example Yeung & Brasseur 1991; Kida & Ohkitani 1992; Ohkitani & Kida 1992; Zhou 1993; Brasseur & Wei 1994) and indeed to the dynamics of any evolving compact vortex. In general the dynamics of transfer can be diverse and very relevant. For example the tendency of two-dimensional vortices to organize surrounding vorticity leads to spiral wind-up and to the decrease of the scales of vorticity gradients. The same phenomenon leads eventually to the formation of larger scales of vorticity. On the other hand, three-dimensional instabilities of filaments of vorticity generally produce smaller vorticity structures.

In this paper we study the interscale transfer properties of the leading-order terms in the Lundgren family of two-dimensional solutions. Specifically we study the nature of interscale energy transfer in the $O(1)$ terms which satisfy the axisymmetric passive differential rotation equation

$$\frac{\partial \omega}{\partial t} + \Omega(r)\,\frac{\partial \omega}{\partial \theta} = 0. \tag{3}$$

This is the equation of passive scalar advection by an axisymmetric azimuthal velocity field, and we generalize our approach to the study of passive scalar advection by a non-axisymmetric azimuthal velocity field, which is governed by the equation

$$\frac{\partial \omega}{\partial t} + \Omega(r,\theta)\,\frac{\partial \omega}{\partial \theta} = 0, \tag{4}$$

where ω stands for passive scalar and $r\Omega(r,\theta)$ is the non-axisymmetric azimuthal velocity.

The mathematical framework and method for the study of interscale transfer in two-dimensional compact vortices that we develop in this paper does not seem to be well suited to the analysis of radial advection which is therefore not dealt with here. It should be stressed that the family of solutions studied here has a certain degree of generality. The functional dependences of the vorticity ω and streamfunction ψ on r and $\theta - \Omega(r)\,t$ are not specified. What is specified is that the explicit dependences on the time t are series of non-negative powers of t^{-1} and that ω is $O(1)$. A lot of the results presented here on interscale transfer in this family of two-dimensional compact vortices are valid irrespective of the specific forms of ω and ψ and are in this sense quite general. However, we also present results for specific choices of ω and ψ for the sake of more detailed illustrative insight into interscale energy transfer in specific two-dimensional compact vortices.

In the following section we describe the mathematical framework that we introduce for the study of energy transfer between different scales of motion in two-dimensional compact vortices. In §§ 3 and 4 we present the study of transfer properties in an axisymmetric azimuthal velocity field. These transfer properties are linear and a consequence of the axisymmetric differential circular advection by the imposed azimuthal velocity $r\Omega(r)$, equation (3). In § 5 the technique is generalized to non-axisymmetric azimuthal advection. We conclude in § 6.

2. Mathematical framework

In the case of a compact structure, a field $\omega(r,\theta)$ in polar coordinates and its Fourier transform $\widehat{\omega}(k,\varphi_k)$, where the wave-vector $\boldsymbol{k} = k\,[\cos\varphi_k, \sin\varphi_k]$, can both be expressed as Fourier series

$$\omega(r,\theta) = \sum_n \omega_n(r)\,\mathrm{e}^{\mathrm{i}n\theta}, \qquad \widehat{\omega}(k,\varphi_k) = \sum_n \widehat{\omega}_n(k)\,\mathrm{e}^{\mathrm{i}n\varphi_k}, \tag{5}$$

with the subscript n corresponding to azimuthal modes. A few standard manipulations lead to (Sneddon 1974; Gilbert 1988)

$$\widehat{\omega}_n(k) = \frac{1}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}(-\mathrm{i})^n\,\omega(r,\theta)\,\mathrm{e}^{-\mathrm{i}\,n\theta}J_n(kr)\,r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{6}$$

Equation (6) shows that a two-dimensional Fourier transform in polar coordinates becomes a coupled azimuthal Fourier series and radial Hankel transform of corresponding order (Sneddon 1974), i.e. the nth Fourier harmonic, $\widehat{\omega}_n(k)$, is the nth-order Hankel transform of the nth Fourier harmonic $\omega_n(r)$

$$\left.\begin{aligned} \omega_n(r) &= 2\pi\mathrm{i}^n\int_0^\infty \widehat{\omega}_n(k)J_n(kr)\,k\,\mathrm{d}k, \\ \widehat{\omega}_n(k) &= \frac{(-\mathrm{i})^n}{2\pi}\int_0^\infty \omega_n(r)J_n(kr)\,r\,\mathrm{d}r. \end{aligned}\right\} \tag{7}$$

For $\omega(r,\theta)$ to be real it must be that $\omega_{-n}(r) = \omega_n^*(r)$, where the asterisk denotes complex conjugate, and it then follows that $\widehat{\omega}_{-n}(k) = (-1)^n\,\widehat{\omega}_n^*(k)$.

From a mathematical point of view, the object of this work is to write the evolution equation for the quantity $\widehat{\omega}_n(k)$ and to analyse the coupling between different wavenumbers.

3. Axisymmetric passive differential rotation

In this section we consider a passive scalar or vorticity field which does not diffuse and is advected by an axisymmetric azimuthal velocity $r\,\Omega(r)$ so that it obeys equation (3). Standard representation in Fourier space with the aid of equations (5) gives

$$\sum_n \frac{\partial\widehat{\omega}_n(k)}{\partial t}\mathrm{e}^{\mathrm{i}n\varphi_k} + \mathrm{i}m\int_0^\infty\int_0^{2\pi}\widehat{\Omega}(|\boldsymbol{k}-\boldsymbol{p}|)\sum_m \widehat{\omega}_m(p)\,\mathrm{e}^{\mathrm{i}m\varphi_p}p\,\mathrm{d}\varphi_p\,\mathrm{d}p\, = 0,$$

where the zeroth-order Hankel transform of the axisymmetric rotation field is (see (7))

$$\widehat{\Omega}(k) = \frac{1}{2\pi}\int_0^\infty \Omega(r)J_0(kr)\,r\,\mathrm{d}r. \tag{8}$$

Introduction of the geometrical equality

$$|\boldsymbol{k}-\boldsymbol{p}| = (k^2 + p^2 - 2kp\,\cos(\varphi_p - \varphi_k))^{1/2} \tag{9}$$

and the change of variables $\beta = \varphi_p - \varphi_k$ eliminates the φ_k-dependence leading to the final form

$$\frac{\partial \widehat{\omega}_n(k)}{\partial t} + \mathrm{i}n \int_0^\infty \widehat{\omega}_n(p)\, \mathscr{A}_n(k,p) p\,\mathrm{d}p = 0, \tag{10}$$

where the transfer kernel $\mathscr{A}_n(k,p)$ is defined as

$$\mathscr{A}_n(k,p) = \int_0^{2\pi} \widehat{\Omega}((k^2+p^2-2kp\cos\beta)^{1/2})\,\mathrm{e}^{\mathrm{i}n\beta}\,\mathrm{d}\beta. \tag{11a}$$

It is worth noting that the vorticity at wavevector $\boldsymbol{k}$ varies in time as a consequence of triadic interactions between $\boldsymbol{k}$, $\boldsymbol{k}-\boldsymbol{p}$ and $\boldsymbol{p}$ for all wavevectors $\boldsymbol{p}$, and that β is the angle between $\boldsymbol{k}$ and $\boldsymbol{p}$. The transfer kernel is an integral over this angle β, each value of β corresponding to a differently shaped triad with the length of one side equal to $k = |\boldsymbol{k}|$ and that of the other equal to $p = |\boldsymbol{p}|$.

The symmetric function $\mathscr{A}_n(k,p) = \mathscr{A}_n(p,k)$ appearing in (10) plays the role of a transfer kernel and represents the filter through which the field at wavenumber p influences its evolution at wavenumber k. This formalism gives an alternative way to look at interaction*s* between scales in compact structures. The transfer kernel couples wavenumbers p and k of the advected field ω by integrating over all differently shaped triads corresponding to different triadic interactions via the driving field Ω which, in general, has energy distributed over all wavenumbers.

An alternative formula for the transfer kernel can be obtained by a more direct approach. From (3)

$$\frac{\partial \widehat{\omega}_n(k)}{\partial t} = \frac{(-\mathrm{i})^n}{2\pi} \int_0^\infty \frac{\partial \omega_n(r)}{\partial t}\, r\, J_n(kr)\,\mathrm{d}r;$$

inserting the equation of motion $\partial\omega_n/\partial t = -\mathrm{i}n\Omega(r)\omega_n(r)$ and expressing $\omega_n(r)$ in terms of its nth-order Hankel transform we finally get

$$\frac{\partial \widehat{\omega}_n(k)}{\partial t} = -\mathrm{i}n \int_0^\infty \widehat{\omega}_n(p) \int_0^\infty \Omega(r)\, J_n(pr)\, J_n(kr)\, r\,\mathrm{d}r\, p\,\mathrm{d}p = 0,$$

which is equivalent to (10) and gives an alternative mathematical definition for the transfer kernel

$$\mathscr{A}_n(k,p) = \int_0^\infty \Omega(r)\, J_n(pr)\, J_n(kr)\, r\,\mathrm{d}r. \tag{11b}$$

Formula (11*a*) has a clearer geometrical or physical interpretation; however it requires the intermediate passage of evaluating $\widehat{\Omega}(k)$. Formula (11*b*) allows in some cases an easier analytical evaluation of the integral.

The rate of energy transfer can be evaluated from the transfer kernel: when $\omega(r,\theta)$ is the vorticity field the energy spectrum is given by (Gilbert 1988)

$$E(k) = \frac{4\pi^3}{k} \left\{ \omega_0^2(k) + 2\sum_{n=0}^{\infty} \widehat{\omega}_n(k)\,\widehat{\omega}_n^*(k) \right\}, \tag{12}$$

where the $*$ superscript stands for complex conjugate. Noticing that

$$\frac{\partial}{\partial t}\left(\widehat{\omega}_n(k)\,\widehat{\omega}_n^*(k)\right) = 2\,\mathrm{Re}\left(\mathrm{i}n\,\widehat{\omega}_n^*(k) \int_0^\infty \widehat{\omega}_n(p)\, \mathscr{A}_n(k,p)\, p\,\mathrm{d}p\right),$$

where equation (10) has been used and Re stands for the real part, then we may write the energy equation

$$\frac{\partial E(k)}{\partial t} + \int_0^{\infty} T(k,p)\,\mathrm{d}p = 0, \tag{13}$$

where the rate of energy transfer $T(k,p)$ is given by

$$T(k,p) = 16\pi^3 \frac{p}{k} \sum_{n=1}^{\infty} n\,\mathscr{A}_n(k,p)\mathrm{Re}\left(\mathrm{i}\widehat{\omega}_n^*(k)\,\widehat{\omega}_n(p)\right), \tag{14}$$

and characterizes the transfer of energy from the wavenumber shell p to the wavenumber shell k. When the field ω is a passive scalar the same procedure applies, with the only difference that its power spectrum is obtained by multiplying by k^2 the right-hand side of equation (12).

The function $T(k,p)$ represents the actual energy transfer once the advecting and advected fields are specified; by contrast the transfer kernel is a characteristic of the advecting field only, and describes how advection provokes interactions between scales. The transfer kernel $\mathscr{A}_n(k,p)$ encapsulates what drives the transfer between scales and its study permits one to understand phenomena involving scale interactions. It is a characteristic of the advecting field independently of the advected one. The next section is devoted to its analysis.

4. Transfer kernel of axisymmetric passive differential advection

4.1. *Algebraic axisymmetric advection*

Let us analyse the transfer kernel $\mathscr{A}_n(k,p)$ corresponding to a differential rotation $\Omega(r) = r^{-1/\alpha}$. The same field has been used by Flohr & Vassilicos (1997) and Angilella & Vassilicos (1999) in their studies of scalar and vorticity dissipation and by Gilbert (1988) to estimate the energy spectrum in two-dimensional vortex fields.

The Hankel transform (8) is given by (Gradshteyn & Ryzhik 1965)

$$\widehat{\Omega}(k) = \frac{\Gamma\left(\dfrac{2\alpha-1}{2\alpha}\right)}{2^{1/\alpha}\pi\Gamma\left(\dfrac{1}{2\alpha}\right)} k^{(1-2\alpha)/\alpha}, \tag{15}$$

where $\Gamma()$ is the gamma function. The result (15) is valid only for $\frac{1}{2} < \alpha < 2$. The lower bound is necessary for the Hankel transform (8) not to diverge at the origin, $r = 0$; the upper bound is necessary for the Hankel transform (8) not to diverge at infinity, $r \to \infty$. It is important to remark that this rotation field Ω is not a large-scale structure; it is a singular compact structure which has energy distributed over all wavenumbers.

Use of formula (11*b*) gives

$$\begin{aligned}\mathscr{A}_n(k,p;\alpha) = {} & \frac{\Gamma\left(n+1-\dfrac{1}{2\alpha}\right)}{2^{(1-\alpha)/\alpha}\Gamma\left(\dfrac{1}{2\alpha}\right)\Gamma(n+1)}(kp)^{1/2\alpha-1}(s+s^{-1}+2)^{1/2\alpha-n-1} \\ & \times F\left(n+1-\frac{1}{2\alpha}, n+\frac{1}{2}; 2n+1; \frac{4}{\left(s+s^{-1}+2\right)}\right),\end{aligned} \tag{16}$$

where $s = k/p$ and $F(a, b; c; d)$ is the Gauss hypergeometric series (Gradshteyn & Ryzhik 1965, 6.576-2; Abramowitz & Stegun 1965). Alternatively, inserting (15) in (11a) gives

$$\mathscr{A}_n(k, p; \alpha) = \frac{\Gamma\left(\dfrac{2\alpha - 1}{2\alpha}\right)}{2^{1/\alpha}\pi\Gamma\left(\dfrac{1}{2\alpha}\right)} (kp)^{(1-2\alpha)/2\alpha} \mathscr{B}_n\left(\frac{k}{p}; \alpha\right), \tag{17}$$

with

$$\mathscr{B}_n(s; \alpha) = \int_0^{2\pi} \frac{e^{in\beta}}{\left(s + 1/s - 2\cos\beta\right)^{(2\alpha-1)/2\alpha}} \, d\beta. \tag{18}$$

The normalized transfer kernel $\mathscr{B}_n(s; \alpha)$ contains the information about the coupling between different scales as a function of the scale ratio. It is more easily integrable numerically than the hypergeometric function in (16) and is of easier interpretation. Because of symmetry $\mathscr{B}_n(s; \alpha) = \mathscr{B}_n(1/s; \alpha)$, and it is therefore sufficient to study (18) in the interval $(1, \infty)$. Note also that $\mathscr{B}_n$ is a real function because of parity. Finally, attention must be drawn to the fact that the integrand in $\mathscr{B}_n(s; \alpha)$ can be singular only when $s = 1$ and that this singularity is at $\beta = 0, 2\pi$ thus indicating the importance of local transfer at $k = p$ by distant triadic interactions (when β is close to 0 or 2π and $k = p$, $|\boldsymbol{k} - \boldsymbol{p}| \ll k$ which is a signature of a distant triad – see Brasseur & Wei 1994).

Equation (18) can be integrated exactly (Gradshteyn & Ryzhik 1965) only for $\alpha = 1$ and $n = 0$. In general it can be integrated numerically, with some care when $s \to 1$ because of the singularity at $s = 1$. Based on our numerical results the asymptotic behaviour has been estimated to be

$$\mathscr{B}_n(s; \alpha) \simeq 2\pi \left(\frac{2\alpha - 1}{2\alpha}\right)^{\sqrt{n}} s^{-\left(n + (2\alpha-1)/2\alpha\right)} \quad \text{as } s \to \infty. \tag{19}$$

It is not easy to obtain this result from asymptotics of (16). This approximation is excellent for $\alpha > 1$, and slightly less accurate (in the coefficient not in the scaling) for $\frac{1}{2} < \alpha < 1$.

Numerical integrations of the normalized transfer kernel $\mathscr{B}_n(s; \alpha)$ are reported in figure 1 for $\alpha = 1.8$, and in figure 2 for $\alpha = 0.6$, where they are plotted against the asymptotic approximation (19). The normalized transfer kernel is extremely peaked at $s = 1$, i.e. $k = p$, indicating that the transfer is essentially local in the sense that the major contribution to the evolution of $\widehat{\omega}_n(k)$ comes from wavenumbers p close to k (see (10), (17), (18) and figure 1).

The asymptotic behaviour when $s \to 1$ is relevant because the normalized transfer kernel assumes larger values and dominates the transfer. Different behaviours are observable when α is larger or smaller than 1. When $s \to 1$ the integrand in (18) becomes increasingly peaked about the points $\beta = 0, 2\pi$, and these points dominate the integral or at least its scaling. In other words the normalized transfer kernel is dominated by distant triads ($\beta = 0, 2\pi$) when $s \to 1$, and a formal integration leads to a scaling $\sim \int (s-1)^{(1-2\alpha)/\alpha}$ resulting in the following asymptotic behaviours as $s \to 1$:

$$\left.\begin{aligned}
&\mathscr{B}_n(s; \alpha) \sim |s - 1|^{(1-\alpha)/\alpha}, && \alpha > 1; \\
&\mathscr{B}_n(s; \alpha) = \begin{cases} -2\log(|s-1|/8), & n = 0, \\ -2\log\left(n|s-1|\right), & n > 0, \end{cases} && \alpha = 1; \\
&\mathscr{B}_n(s; \alpha) \sim \text{finite value}, && \alpha < 1.
\end{aligned}\right\} \tag{20}$$

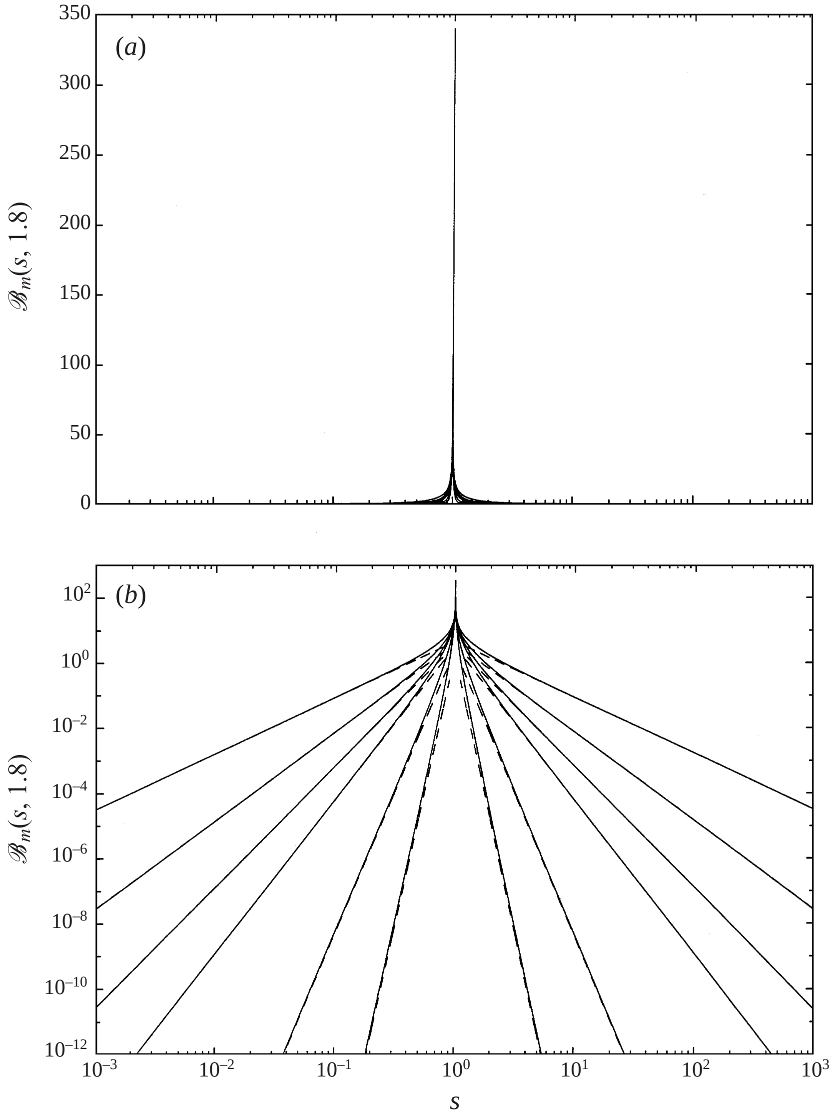

FIGURE 1. Normalized transfer kernel for an algebraic axisymmetric advection, $\mathscr{B}_m(s)$, defined by (18), for $\alpha = 1.8$ and $m = 1, 2, 3, 4, 8, 16$ (from higher to lower values, respectively). The dashed lines represent asymptotic behaviour (19). Semi-logarithmic representation (*a*), bilogarithmic (*b*).

The result for $\alpha = 1$ can be found analytically; the actual limit value for $\alpha < 1$ can be obtained from (16) evaluated at $k = p$. These asymptotic behaviours near $s = 1$ are plotted in figure 3 where it is seen that the numerical results confirm the asymptotic approximations (20). The normalized transfer kernel is power-law singular when $\alpha > 1$, and is not singular when $\alpha < 1$. The singular behaviour at $\alpha > 1$ reflects the unboundness of the azimuthal velocity $u_\theta = r\,\Omega(r) \sim r^{(\alpha-1)/\alpha}$ as $r \to \infty$.

The well-defined sharp peak of the transfer kernel at $k = p$ indicates that the transfer is local in the sense that every wavenumber k evolves, see (10), excited via the differential rotation by itself and neighbouring wavenumbers only. This excitation occurs via the differential rotation and more specifically mostly via distant triadic

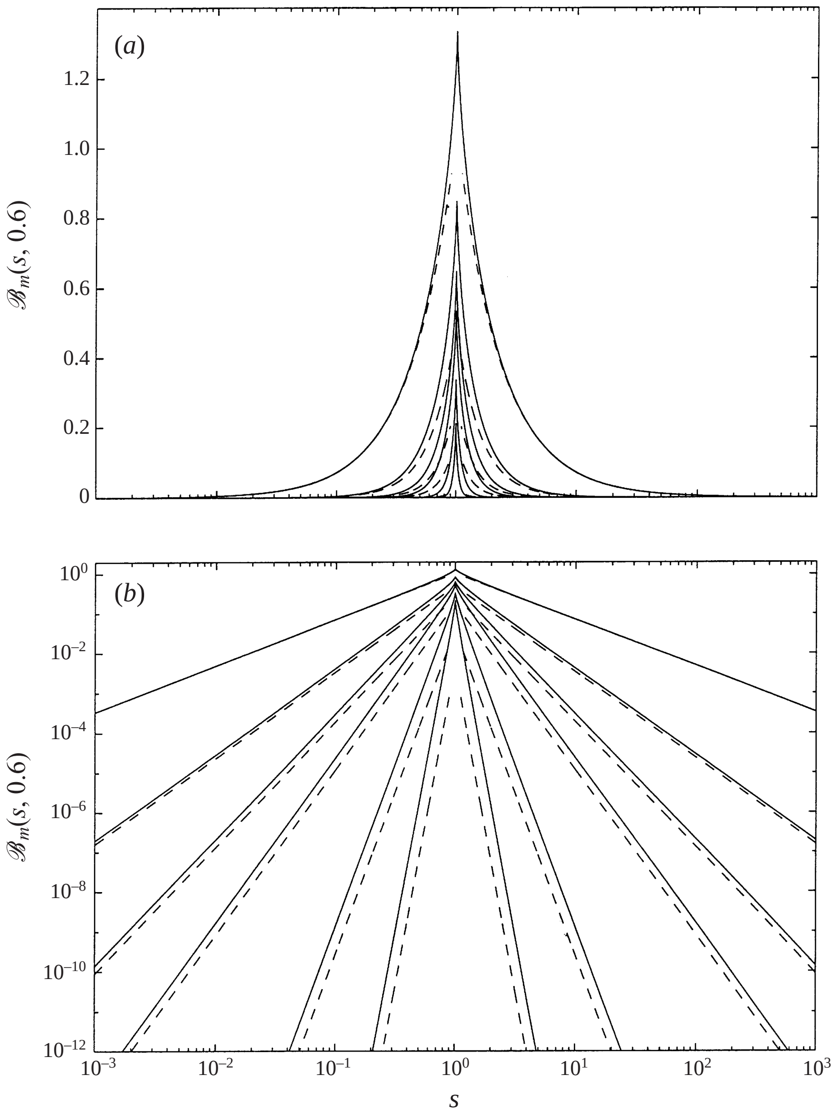

FIGURE 2. Like figure 1 but for $\alpha = 0.6$.

interactions because the transfer kernel at $k = p$ is dominated by the contributions of β around 0 and 2π. Finally, in the evolution of the field $\widehat{\omega}_n(k)$ every mode n depends just on itself (axisymmetric advection in the azimuthal direction).

The locality becomes increasingly sharp with increasing α (see (20) and figures 1, 2, and 3) corresponding to a decrease in steepness of the differential rotation's profile, and therefore to a more uniform local shear. Interactions between scales are widened when the differential rotation is steeper thus suggesting that delocalization is induced by the non-uniformity of shear. This important conclusion can be directly understood by inspection of equation (11*a*): the transfer kernel between two scales k and p is given by integration of the transformed rotation $\widehat{\Omega}$ in the range between $|k - p|$ and $(k + p)$. When the scales k and p are well separated this range is very narrow about the largest of k and p, and only a highly non-uniform shear (differential

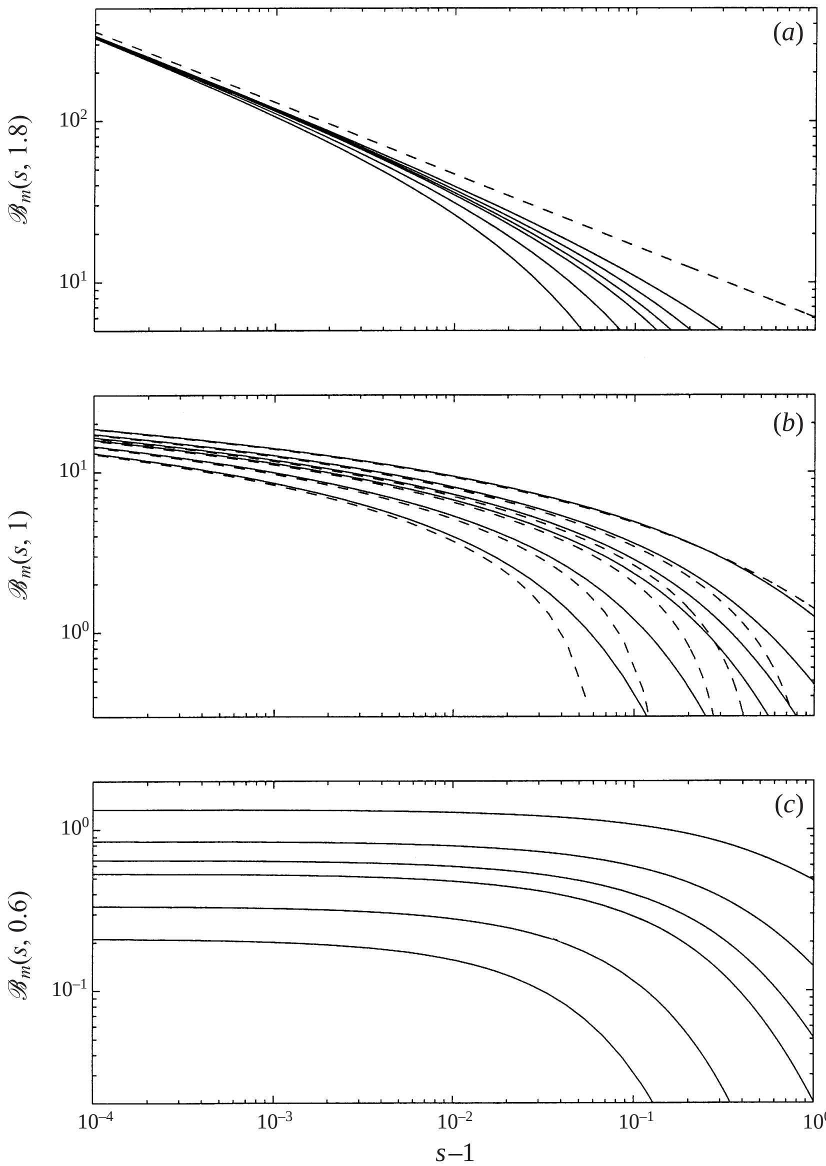

FIGURE 3. Representation near $s = 1$ of the normalized transfer kernel for an algebraic axisymmetric advection, $\mathscr{B}_m(s)$, defined by (18), for $\alpha = 1.8$ (*a*), $\alpha = 1$ (*b*), $\alpha = 0.6$ (*c*), and $m = 1, 2, 3, 4, 8, 16$ (from higher to lower values, respectively). The dashed lines represent asymptotics (20).

rotation) about such a scale can give a non-zero integral and thereby lead to a direct interaction between the wavenumbers k and p. Conversely, when the scales k and p are comparable, even a slow variation of shear over the whole range from about zero to $k + p$ is enough to produce a significant local interaction between k and p.

We may now draw the following physical picture which we confirm in the following sections. Transfer is produced by the presence of differential rotation (shear). The locality of the transfer kernel is a consequence of the uniformity of shear; and only the presence of a variation of shear over a certain scale can generate a non-local interaction between this and other scales in the evolution of $\widehat{\omega}_n(k)$, equation (10). As

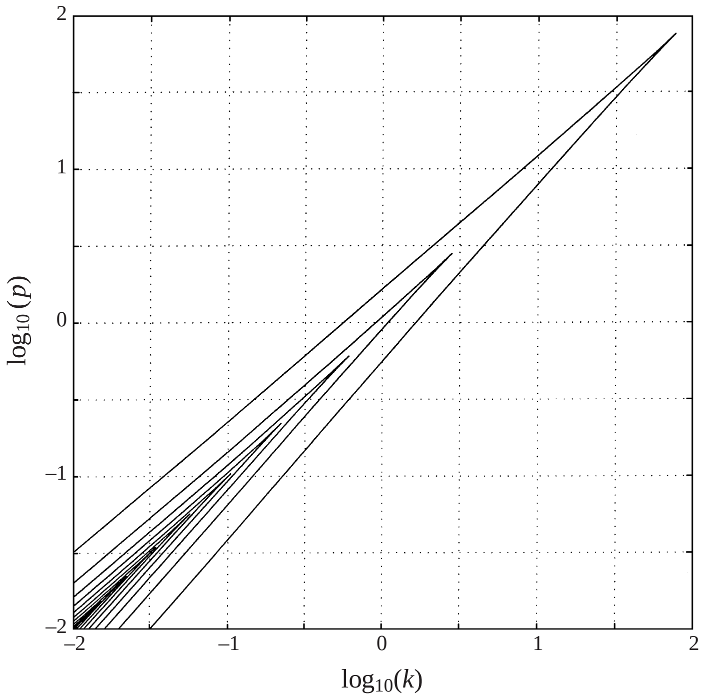

FIGURE 4. Contour plot of the transfer kernel for an algebraic axisymmetric advection, $\mathscr{A}_m(k,p)$, defined by (17), for $\alpha = 0.6$ and $m = 2$. Outer to inner levels from 0.2 to 8.2, step 0.4.

a consequence, the transfer kernel is more local at increasing wavenumbers. Indeed, these are associated with smaller physical scales over which the shear is increasingly uniform. The locality of the transfer kernel is also more pronounced for higher modes n, as depicted in figures 1(b) and 2(b). This is due to the particular character of the Fourier–Hankel transform where azimuthal and radial transformations are not totally decoupled and high modes n correspond to high wavenumbers k. This relation between high modes n and high wavenumbers k follows from the behaviour of the Bessel function in (7b). The complete two-dimensional shape of the transfer kernel $\mathscr{A}_n(k,p)$ can be obtained from equation (17); it is reported in figure 4 for $\alpha = 0.6$ and $n = 2$, and confirms the increase of transfer kernel locality at high wavenumbers.

The study of the transfer kernel is attractive because it is general in the sense that it applies to any form of the advected field ω. It is also quite far-reaching in the sense that a good understanding of the transfer kernel is enough to derive from (10) the transfer properties within ω from one mode $\widehat{\omega}_n(k)$ to another. Different advected fields ω correspond to different initial conditions for the solution of (10) and the transfer kernel's transfer properties are therefore in this sense independent of initial conditions. In particular we have learned that the locality of the transfer kernel is dominated by distant triads and this property is independent of the initial form of ω. However, a complete picture of transfer should also include a description of energy transfer which cannot be given without specifying the advected field ω, see equation (14). What can, however, be said about the rate of energy transfer independently of the choice of ω and by direct inspection of (14) is that $T(k,k) = 0$.

We calculate $T(k,p)$, from (14), for a single spiral vortex sheet form of ω (see

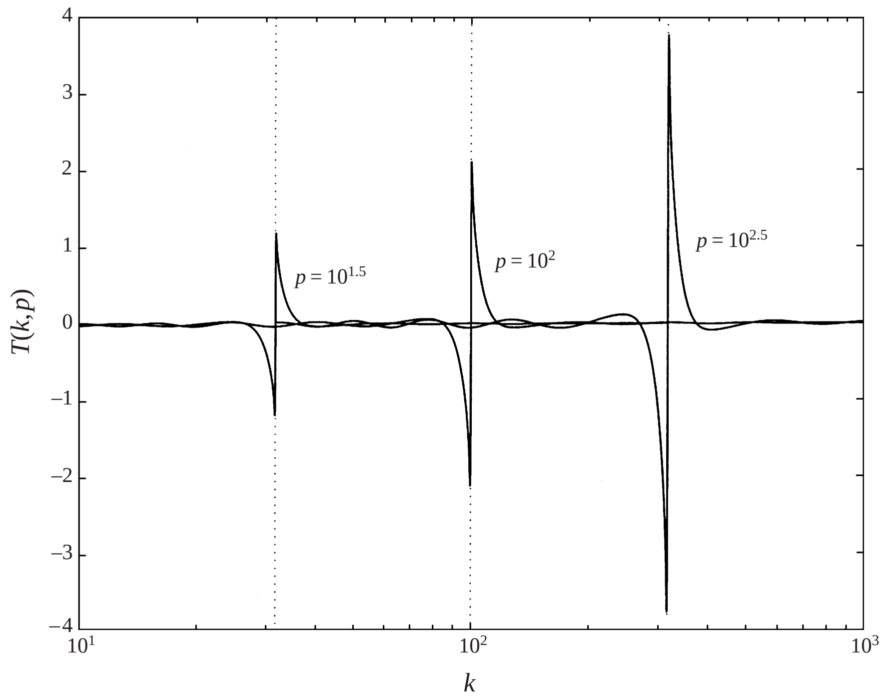

FIGURE 5. Transfer spectrum $T(k,p)$ for an algebraic axisymmetric advection with $\alpha = 0.6$ of a spiral vortex sheet, as a function of k for $p = 10^{1.5}, 10^2, 10^{2.5}$.

equations (A 1), (A 2) and (A 5) in Vassilicos & Brasseur 1996 where the specific form of a single two-dimensional Lundgren spiral vortex sheet ω is given) and $\Omega(r) = r^{-1/\alpha}$.

The Fourier series – Hankel transform (6), (7) of the spiral vorticity field cannot be evaluated analytically, but an approximation can be obtained for large wavenumber k. For large kr we can approximate the Bessel function by its asymptotic form and the resulting integral is then approximated by the method of stationary phase to eventually give an analytical expression for $\widehat{\omega}_n(k)$ (neglecting the contribution of the singularity of $\omega(r,\theta)$ at the centre, on the basis of the results given in the following §4.2). The rate of transfer spectrum is then computed by equation (14) using the numerically evaluated transfer kernel presented above in this section.

In figure 5 we report a numerical calculation of $T(k,p)$ for $\alpha = 0.6$. Following Yeung & Brasseur (1991) we plot $T(k,p)$ as a function of k for different logarithmically spaced values of p. The shape of the functional dependence of $T(k,p)$ on k is the same for the three values of p because of the self-similarity of the vortex structure. We find that $T(k,p)$ is positive for $k > p$ and negative for $k < p$ which indicates an energy flux from large to small lengthscales. Also, $T(k,p)$ is peaked very close to $k = p$ on both sides of $k = p$, thereby indicating locality of energy transfer in wavenumber space, and the intensity of this peak increases with wavenumber. This behaviour seems qualitatively analogous to that observed by Yeung & Brasseur (1991) in three-dimensional homogeneous isotropic turbulence (see their figure 4) where $T(k,p)$ has the same qualitative signature but with a locality of energy transfer that is not so sharp.

4.2. *Smoothed and bounded axisymmetric advection*

The results of the previous subsection have been derived for a singular and unbounded differential rotation. To ascertain the general validity of our physical conclusions it is necessary to verify them when the advecting field is everywhere regular.

Let us begin by considering the algebraic rotation of the previous section but with the extra care of smoothing it out close to the origin to avoid the presence of a singularity at $r = 0$. Let us consider a *core* radius ρ such that

$$\Omega(r) = \begin{cases} \Omega_\rho, & r \leqslant \rho, \\ \Omega_\rho \left(\dfrac{r}{\rho}\right)^{-1/\alpha}, & \rho \leqslant r. \end{cases} \tag{21}$$

Recalling the relations

$$\kappa(r) = 2\pi r^2 \Omega(r), \qquad \frac{\mathrm{d}\kappa}{\mathrm{d}r} = 2\pi r \gamma(r),$$

where $\kappa(r)$ is the circulation and $\gamma(r)$ is the azimuthally averaged vorticity, then we have

$$\frac{\kappa(r)}{2\pi} = \begin{cases} \Omega_\rho r^2, & r \leqslant \rho, \\ \Omega_\rho r^2 \left(\dfrac{r}{\rho}\right)^{-1/\alpha}, & \rho \leqslant r; \end{cases} \qquad \gamma(r) = \begin{cases} 2\,\Omega_\rho, & r \leqslant \rho, \\ \dfrac{2\alpha - 1}{\alpha} \Omega_\rho \left(\dfrac{r}{\rho}\right)^{-1/\alpha}, & \rho \leqslant r. \end{cases}$$

It must be noticed that, physically, this smoothed field corresponds to substituting the singular vorticity field $\gamma(r) = [(2\alpha - 1)/\alpha] r^{-1/\alpha}$ inside the core radius ρ with a constant vorticity field having the same total circulation. However this substitution is only meaningful for $\alpha > \frac{1}{2}$ otherwise the circulation is infinite inside the core (non-integrable singularity). The smoothed field (21) can nevertheless be considered even when $\alpha \leqslant \frac{1}{2}$ in which case it should be kept in mind that the constant-vorticity core does not derive from smoothing of a singular field. In the case where the differential rotation is induced by a spiral vortex sheet the core size should increase with time because of viscous dissipation, but this time dependence would only contribute a time dependence to the transfer kernel without affecting the properties of the transfer itself which derives from instantaneous pictures of the advecting field.

The Hankel transform (8) of the differential rotation (21) can be written in dimensionless form (i.e. $\rho = 1$, $\Omega_\rho = 1$) as follows:

$$\widehat{\Omega}(k) = \frac{1}{2\pi} \left\{ \int_0^1 J_0(kr)\, r\, \mathrm{d}r + \int_0^\infty r^{-1/\alpha} J_0(kr)\, r\, \mathrm{d}r - \int_0^1 r^{-1/\alpha} J_0(kr)\, r\, \mathrm{d}r \right\}. \tag{22}$$

Let us examine the three contributions separately. The first integral

$$\int_0^1 J_0(kr)\, r\, \mathrm{d}r = k^{-1} J_1(k) \sim \begin{cases} \text{const}, & k \ll 1, \\ k^{-3/2}, & k \gg 1; \end{cases} \tag{23a}$$

the second is given by (15)

$$\int_0^\infty r^{-1/\alpha} J_0(kr)\, r\, \mathrm{d}r = \frac{2^{(\alpha-1)/\alpha} \Gamma\left(\dfrac{2\alpha - 1}{2\alpha}\right)}{\Gamma\left(\dfrac{1}{2\alpha}\right)} k^{(1-2\alpha)/\alpha} \sim k^{(1-2\alpha)/\alpha}; \tag{23b}$$

and the third integral can be written in terms of products of Bessel and Struve functions (Gradshteyn & Ryzhik 1965) which can also be written as (Abramowitz & Stegun 1965)

$$\int_0^1 r^{-1/\alpha} J_0(kr)\, r\, \mathrm{d}r = \frac{\Gamma\left(\dfrac{2\alpha-1}{2\alpha}\right)}{\Gamma\left(\dfrac{1}{2\alpha}\right)} k^{-1} \sum_{n=0}^{\infty} \frac{(2n+1)\,\Gamma\left(\dfrac{1}{2\alpha}+n\right)}{\Gamma\left(2-\dfrac{1}{2\alpha}+n\right)} J_{2n+1}(k)$$

$$\sim \begin{cases} \text{const}, & k \ll 1, \\ k^{(1-2\alpha)/\alpha}, & k \gg 1. \end{cases} \qquad (23c)$$

The second contribution (23b) dominates at small wavenumbers; at large wavenumbers the sum of contributions (23b) and (23c), if considered with their exact coefficients (which are oscillatory functions of k), produce cancellations of the $k^{(1-2\alpha)/\alpha}$ terms leading to a $k^{-3/2}$ scaling which combines with (23a) to give a steeper $-\frac{5}{2}$ power law; thus

$$\widehat{\Omega}(k) \sim \begin{cases} k^{(1-2\alpha)/\alpha}, & k \ll 1, \\ k^{-5/2}, & k \gg 1. \end{cases} \qquad (24)$$

This result has been verified numerically with particular care taken in evaluating the summation in (23c). The transfer kernel corresponding to the smoothed differential rotation (21) has been evaluated numerically on the basis of (22) and (23) and confirms the results concerning the locality of transfer kernel obtained in the previous subsection for the unsmoothed case. In fact this locality is determined by the small-wavenumber behaviour of $\widehat{\Omega}(k)$ which is not modified by smoothing. Non-local transfer is as before a minor contribution to equation (10) but now decays even faster with s because of the faster decay of $\widehat{\Omega}(k)$ at large wavenumbers. The smoothing at the origin eliminates variations of shear over the smallest length-scales and thereby reduces non-local interactions between scales.

Because the small-wavenumber limit is relevant to the properties of transfer between comparable wavenumbers, we now consider a smoothed differential rotation corresponding to zero vorticity beyond a finite radial location. Normalizing with this external radius and the rotation there we consider the rotation field

$$\Omega(r) = \begin{cases} \rho^{-1/\alpha}, & r \leqslant \rho, \\ r^{-1/\alpha}, & \rho \leqslant r \leqslant 1, \\ r^{-2}, & 1 \leqslant r, \end{cases} \qquad (25)$$

with Hankel transform (8) given by

$$\widehat{\Omega}(k) = \frac{1}{2\pi}\left\{ \rho^{-1/\alpha} \int_0^{\rho} J_0(kr)\, r\, \mathrm{d}r + \int_{\rho}^{1} r^{-1/\alpha} J_0(kr)\, r\, \mathrm{d}r + \int_1^{\infty} r^{-2} J_0(kr)\, r\, \mathrm{d}r \right\}. \qquad (26)$$

The first two integrals can be evaluated from formulae (23a) and (23c), respectively. These two integrals dominate the large-wavenumber behaviour and imply that the same scaling (24) is valid for the bounded field (26) at small scales ($k \gg \rho^{-1} > 1$). In the limit of small wavenumbers the first two integrals tend to a constant. The last integral in (26) can be expressed either in terms of a generalized hypergeometric function (namely ${}_2F_3$) or as

$$\int_1^{\infty} r^{-2} J_0(kr)\, r\, \mathrm{d}r = -\gamma - \log\frac{k}{2} - \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{4}k^2\right)^n}{2n\,(n!)^2} \underset{k\to 0}{\sim\!\longrightarrow} -\log k, \qquad (27)$$

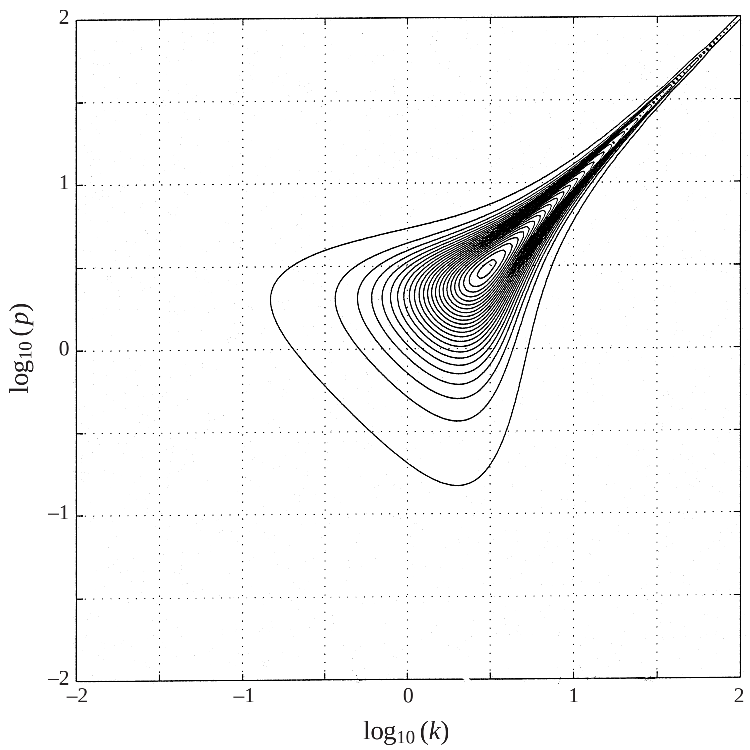

FIGURE 6. Contour plot of the transfer kernel for a Gaussian axisymmetric advection, $\mathscr{A}_m(k,p)$, defined by (28), for $m=2$. Outer to inner levels from 5×10^{-4} to 5.8×10^{-2}, step 2.5×10^{-3}.

and dominates the small-wavenumber limit of $\widehat{\Omega}(k)$ which is logarithmically divergent. This result influences the behaviour of the transfer kernel in the limit $k/p\to1$ and inhibits the power-law singularity (20) which appears when the azimuthal velocity is divergent at $r\to\infty$. The kernel is now non-singular independently of α, a fact that has also been verified numerically. Thus the degree of transfer kernel locality in a bounded field is reduced for $\alpha\geqslant1$ and is about unchanged for $\alpha<1$. However the overall shape of the transfer kernel remains substantially the same in all cases.

A physically relevant and particularly simple case of a regular field is the Gaussian advection $\Omega(r)=\mathrm{e}^{-r^2}$. It corresponds to $\widehat{\Omega}(k)=(4\pi)^{-1}\,\mathrm{e}^{-k^2/4}$ and from either formulae (11) the transfer kernel can be evaluated analytically (Gradshteyn & Ryzhik 1965) as

$$\mathscr{A}_n(k,p)=\frac{1}{2}I_n\left(\frac{kp}{2}\right)\mathrm{e}^{-(k^2+p^2)/4}, \tag{28}$$

where $I_n(\ )$ is the modified Bessel function. As for the previous cases the transfer kernel is peaked at $k=p$ indicating the dominance of local transfer in equation (10) and presents an exponential decay with s. The smoothness at the origin and the exponential decrease of the Gaussian field makes it particularly prone to numerical treatment. The shape of the transfer kernel (28) is reported in figure 6 for $n=2$. At high wavenumbers it is analogous to the singular shape reported in figure 4, even though the decay is now exponential rather than algebraic. An absence of transfer can be observed at lengthscales larger than the characteristic lengthscale of the Gaussian advection, here of order one. No such lengthscale exists in the case of the self-similar algebraic advection.

These results confirm the conclusion that an axisymmetric advection by a monotonic differential rotation leads to predominantly local interactions between modes $\widehat{\omega}_n(k)$. This locality derives from the relative uniformity of shear over a given scale. Some degree of delocalization can occur at smaller wavenumbers because the shear's variation is more likely to be felt over larger lengthscales; on the other hand locality is more pronounced at higher wavenumbers and also for higher azimuthal modes, which are associated with smaller lengthscales, because relative to them any dominant large-scale shear appears as uniform.

The connection between locality and uniformity of shear would also imply that non-local interactions can occur when the azimuthal velocity presents radial oscillations with a well-defined wavelength λ. In this case, the Hankel transform (8) is peaked about λ^{-1} and the transfer kernel $\mathscr{A}_n(k,p)$ is non-zero only where $|k-p| < \lambda^{-1} < (k+p)$ resulting in a non-local interaction. This conclusion has been verified explicitly for some rather academic cases of radially oscillating differential rotations.

4.3. *Non-axisymmetric transfer during wind-up of spiral vortex sheets*

In the case of a spiral vortex sheet the axisymmetric transfer results assume a deeper relevance because of the special form taken by the leading, $O(t^{-1})$, non-axisymmetric terms appearing in equation (2). In fact it is shown below that the sum of such terms is exactly zero and the leading correction is of order t^{-2}.

A general single spiral vorticity field, with azimuthally averaged angular velocity $\Omega(r)$, can be expressed in Fourier series (5) (Lundgren 1982) with each mode expressed in the form

$$\omega_n(r) = f_n(r)\,\mathrm{e}^{-\mathrm{i}n\Omega(r)\,t}, \tag{29}$$

where the $f_n(r)$ are generic $O(1)$ functions. This vorticity field corresponds, in incompressible flow, to streamfunction azimuthal modes (Vassilicos & Brasseur 1996)

$$\psi_n(r) = \frac{f_n(r)}{\left(nt\,\mathrm{d}\Omega/\mathrm{d}r\right)^2}\,\mathrm{e}^{-\mathrm{i}n\Omega(r)t} + o(t^{-2}), \tag{30}$$

from which the velocity components can be obtained.

Let us look at the $O(t^{-1})$ terms in equation (2), namely the third and fourth terms. We want to compare the two nonlinear products in Fourier space

$$\left[\frac{\partial\psi}{\partial r}\right]_n \left[\frac{\partial\omega}{\partial\theta}\right]_m \quad\text{and}\quad \left[\frac{\partial\psi}{\partial\theta}\right]_n \left[\frac{\partial\omega}{\partial r}\right]_m .$$

From (29), (30) it is easy to verify that the first product

$$\left[\frac{\partial\psi}{\partial r}\right]_n \left[\frac{\partial\omega}{\partial\theta}\right]_m = -\frac{f_n(r)f_m(r)}{t\,\mathrm{d}\Omega/\mathrm{d}r}\,\frac{m}{n}\mathrm{e}^{-\mathrm{i}\,(n+m)\,\Omega(r)\,t},$$

is exactly equal and opposite in sign to the second one leading to the conclusion that, in the case of an evolving spiral vortex sheet, the leading correction to the axisymmetric advection is $O(t^{-2})$.

The transfer properties associated with an axisymmetric differential advection have been obtained in the previous sections where the $O(t^{-1})$ terms in the evolution of a general Lundgren vortex were neglected. This subsection's result shows that, when considering the transfer in a Lundgren spiral vortex sheet, the $O(t^{-1})$ terms cancel exactly and the leading correction is $O(t^{-2})$, independently of the specific spiral geometry, rotation and vorticity distribution along the sheet. Hence, in the case of an evolving spiral vortex sheet of the Lundgren type, the conclusions obtained in the previous sections concerning the locality of transfer in an axisymmetric advection field remain correct to a good approximation even when the leading non-axisymmetric terms of the advection field are included.

5. Non-axisymmetric differential rotation of a passive scalar field

5.1. *Convolution of an azimuthally sinusoidal term and a generic field*

When writing the Fourier representation of the equation of motion (4) the main difficulty is in expressing the convolution terms in a form analogous to the linear term. In §§ 3 and 4 we have analysed the case where the advecting field is axisymmetric, i.e. the second term in equation (4). Let us now consider a generic azimuthally sinusoidal advecting field $\Omega(r,\theta) = \Omega_m(r)\mathrm{e}^{\mathrm{i}m\theta}$ and how its product with an arbitrary field $f(r,\theta)$, which represents the θ-derivative of ω, can be written as a convolution in Hankel–Fourier space.

Using the same notation as in § 3 we can write

$$\begin{aligned}\Omega(r,\theta)f(r,\theta) &= \iiiint_{-\infty}^{+\infty} \widehat{\Omega}(\boldsymbol{k}-\boldsymbol{p})\,\widehat{f}(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k}\\ &= \iint_{-\infty}^{+\infty}\left\{\int_0^{\infty}\int_0^{2\pi}\widehat{\Omega}(|\boldsymbol{k}-\boldsymbol{p}|\,,\varphi_{k-p})\,\widehat{f}(p,\varphi_p)\,\mathrm{d}\varphi_p\,p\,\mathrm{d}p\right\}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k}. \qquad (31)\end{aligned}$$

The part inside the braces represents the Fourier transform of the product field and can be expressed as

$$\begin{aligned}&\int_0^{\infty}\int_0^{2\pi}\widehat{\Omega}(|\boldsymbol{k}-\boldsymbol{p}|\,,\varphi_{k-p})\,\widehat{f}(p,\varphi_p)\,\mathrm{d}\varphi_p\,p\,\mathrm{d}p\\ &\qquad = \int_0^{\infty}\sum_{\ell}\widehat{f}_{\ell}(p)\left\{\int_0^{2\pi}\widehat{\Omega}_m(|\boldsymbol{k}-\boldsymbol{p}|)\,\mathrm{e}^{\mathrm{i}m\varphi_{k-p}}\mathrm{e}^{\mathrm{i}\ell\varphi_p}\,\mathrm{d}\varphi_p\right\}p\,\mathrm{d}p. \qquad (32)\end{aligned}$$

It is now necessary to express the modulus and phase of $\boldsymbol{k}-\boldsymbol{p}$ in terms of moduli and phases of $\boldsymbol{k}$ and $\boldsymbol{p}$ separately. The former is given by equation (9); for the latter we must write down the components

$$|\boldsymbol{k}-\boldsymbol{p}|\begin{bmatrix}\cos\varphi_{k-p}\\ \sin\varphi_{k-p}\end{bmatrix} = \begin{bmatrix}k\cos\varphi_k - p\cos\varphi_p\\ k\sin\varphi_k - p\sin\varphi_p\end{bmatrix};$$

and multiplications by $\cos\varphi_p$ and $\sin\varphi_p$ and subsequent summation results in

$$\varphi_{k-p} = \varphi_p + \tan^{-1}\left(\frac{\sin(\varphi_k-\varphi_p)}{\cos(\varphi_k-\varphi_p) - p/k}\right), \qquad (33)$$

where $\tan^{-1}$ is the four quadrants inverse tangent. Equation (33) is very important in what follows and the fact that $\varphi_{k-p}-\varphi_p$ depends on $\varphi_k-\varphi_p$ only and not on φ_k and φ_p separately allows the necessary simplifications.

Using (9) and (33) the part inside the braces in equation (32) can be rewritten as follows:

$$\int_0^{2\pi} \widehat{\Omega}_m\left(\sqrt{k^2+p^2-2kp\cos\left(\varphi_p-\varphi_k\right)}\right)$$
$$\times\exp\left(\mathrm{i}(m+\ell)\varphi_p+\mathrm{i}m\tan^{-1}\left(\frac{\sin\left(\varphi_k-\varphi_p\right)}{\cos\left(\varphi_k-\varphi_p\right)-(p/k)}\right)\right)\mathrm{d}\varphi_p$$
$$=\mathrm{e}^{\mathrm{i}(m+\ell)\varphi_k}\int_0^{2\pi}\widehat{\Omega}_m\left(\sqrt{k^2+p^2-2kp\cos\beta}\right)$$
$$\times\exp\left(\mathrm{i}\,(m+\ell)\,\beta-\mathrm{i}m\tan^{-1}\left(\frac{\sin\beta}{\cos\beta-p/k}\right)\right)\,\mathrm{d}\beta, \tag{34}$$

where the change of variable $\beta=\varphi_p-\varphi_k$ has been used and the periodicity of the functions under the integral sign allowed the shift of the integration interval.

Inserting (34) back in (32), and then (32) back in (31) we can finally write

$$\Omega(r,\theta)f(r,\theta)=\iint_{-\infty}^{+\infty}\sum_n\left\{\int_0^\infty\widehat{f}_{n-m}(p)\mathscr{A}_{m,n}(k,p)\,p\,\mathrm{d}p\right\}\,\mathrm{e}^{\mathrm{i}n\,\varphi_k}\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{k}, \tag{35}$$

where the transfer kernel $\mathscr{A}_{m,n}\,(k,p)$ is given by

$$\mathscr{A}_{m,n}(k,p)=\int_0^{2\pi}\widehat{\Omega}_m\left(\sqrt{k^2+p^2-2kp\cos\beta}\right)$$
$$\times\exp\left(\mathrm{i}\,n\beta-\mathrm{i}m\tan^{-1}\left(\frac{\sin\beta}{\cos\beta-p/k}\right)\right)\mathrm{d}\beta. \tag{36a}$$

This generalized transfer kernel represents coupling in wavenumber space, due to forcing mode m, between azimuthal modes n and $n-m$ and wavenumbers k and p.

By analogy with the procedure in § 3 an alternative formula to (36a) can be given in this case too; starting from the identity

$$\Omega(r,\theta)f(r,\theta)$$
$$=\iint_{-\infty}^{+\infty}\sum_n\left\{\frac{(-\mathrm{i})^n}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}\Omega(r,\theta)f(r,\theta)\,\mathrm{e}^{-\mathrm{i}\,n\,\theta}J_n(kr)\,\mathrm{d}\theta r\,\mathrm{d}r\right\}\,\mathrm{e}^{\mathrm{i}n\,\varphi_k}\mathrm{e}^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k},$$

the term inside the braces corresponds to the term inside braces in (35). Inserting the Fourier–Hankel representation of $f(r,\theta)$ and the azimuthal Fourier representation of $\Omega(r,\theta)$, it can be rewritten as

$$\frac{(-\mathrm{i})^n}{2\pi}\int_0^\infty\int_0^\infty\sum_\ell\mathrm{i}^\ell\,\Omega_m(r)\widehat{f}_\ell(p)J_\ell(pr)J_n(kr)\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(\ell+m-n)\,\theta}\mathrm{d}\theta r\,\mathrm{d}rp\,\mathrm{d}p$$
$$=(-\mathrm{i})^m\int_0^\infty\widehat{f}_{n-m}(p)\left\{\int_0^\infty\Omega_m(r)J_{n-m}(pr)J_n(kr)\,r\,\mathrm{d}r\right\}\,p\,\mathrm{d}p.$$

Comparison with (35) gives the alternative formula

$$\mathscr{A}_{m,n}(k,p)=(-\mathrm{i})^m\int_0^\infty\Omega_m(r)J_{n-m}(pr)J_n(kr)\,r\,\mathrm{d}r, \tag{36b}$$

which is generally better suited for analytical integration whereas (36a) is more suited to numerical integration.

From (36) we derive the symmetry property of the non-axisymmetric transfer kernel

$$\mathscr{A}_{m,n}(k,p) = (-1)^m \mathscr{A}_{m,m-n}(p,k), \tag{37}$$

and the reality condition $\Omega_m(r) = \Omega^*_{-m}(r)$ also gives $\mathscr{A}_{m,n} = \mathscr{A}^*_{-m,-n}$. The symmetry property (37) implies that for some pairs m even and $n = m/2$ the transfer kernel is symmetric in k and p. However the transfer kernel is not in general symmetric and should not be expected to present a maximum at $k = p$ which would reflect a preponderance of local interactions between scales.

5.2. *Application to non-axisymmetric differential rotation*

Let us focus on equation (4)

$$\frac{\partial \omega}{\partial t} + \Omega(r,\theta)\frac{\partial \omega}{\partial \theta} = 0.$$

Here ω is not the vorticity field but a passive scalar field because the relation between vorticity and velocity is not accounted for. Using the Fourier series representation $\Omega(r,\theta) = \sum_m \Omega_m(r)\,\mathrm{e}^{\mathrm{i}m\theta}$, a straightforward application of (35) to (4) gives

$$\frac{\partial \widehat{\omega}_n(k)}{\partial t} + \sum_m \int_0^\infty \{\mathrm{i}\,(n-m)\;\widehat{\omega}_{n-m}(p)\mathscr{A}_{m,n}(k,p)\}\,p\,\mathrm{d}p = 0, \tag{38}$$

with obvious meaning of symbols.

It must be said that the application of this same procedure to radial advection is not straightforward because the Hankel transform of the radial derivative, $[\widehat{\partial \omega/\partial r}]_n(k)$, cannot be expressed in general in terms of $\widehat{\omega}_m(p)$ (Sneddon 1974).

6. Conclusions

We have shown that the transfer between scales in Lundgren two-dimensional evolving compact vortices is predominantly local in wavenumbers and due to distant triadic interactions. Transfer is local at a given lengthscale when the shear (differential rotation) does not vary much over that lengthscale. Local transfer between different modes (equations (10) and (38)) is therefore more pronounced at higher wavenumbers and for higher azimuthal modes, both associated with smaller lengthscales, because any differential rotation appears increasingly uniform over decreasing lengthscales. Non-local interactions can occur at a given scale only if the advecting field presents significant variation of shear over that scale.

A new framework has been developed to describe the transfer between scales inside compact structures which may also find useful applications in other situations. The analysis of the transfer kernel may shed some light on a variety of advection phenomena as well as reveal signatures of instabilities.

Locality of scale interactions has been shown to be a typical phenomenon in axisymmetric advection, especially when the vortex field has a spiral structure. However, in the case of non-axisymmetric advection, azimuthal oscillations imply the presence of physical wavelengths that may give rise to non-local interactions.

We conclude by mentioning that non-local interactions may be expected to be a common phenomenon in non-axisymmetric flows. As a simple example, let us consider the azimuthal advection due to a quadripolar physical structure $\Omega_{\pm 2} = r^2 \mathrm{e}^{-r^2/2}$ whose

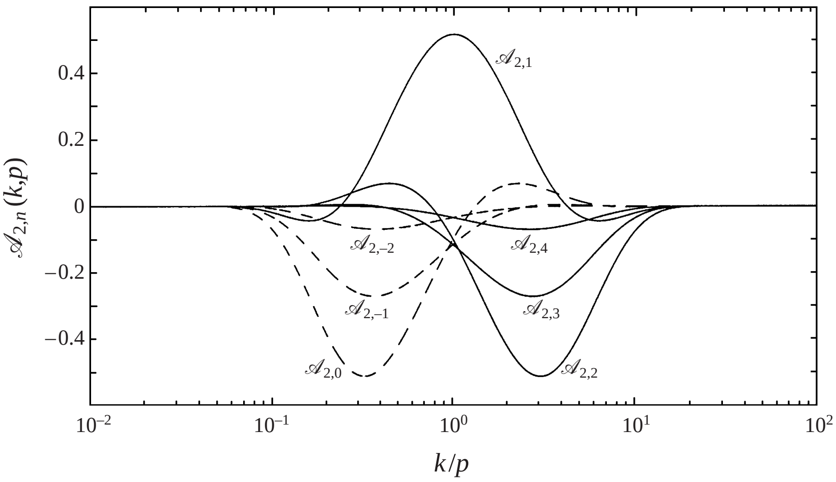

FIGURE 7. Non-axisymmetric $m = 2$ transfer kernel, given by (39), as a function of the ratio k/p for a fixed value $kp = 1$ at various n. The pairs of non-symmetric functions (n pairing with $2 - n$) as given by relations (37), are shown with continuous and dashed lines, respectively.

corresponding transfer kernel is (Prudnikov, Brychkov & Marichev 1986, 2.12.39-4)

$$\mathscr{A}_{2,n}(k,p) = (2-n)\,k^{-2}\left\{kp\left(2-k^2\right)I_{n-1}(kp) - kp\left(2n+k^2\right)I_{n-3}(kp)\right. \\ \left. + \left[4n\,(n-2) + k^2\left(2n-4+p^2\right) + k^4\right]I_{n-2}(kp)\right\}\mathrm{e}^{-(k^2+p^2)/2}. \qquad (39)$$

The function (39) is shown in figure 7, for $kp = 1$, as a function of k/p. The transfer kernel of this field, which may represent a $m = 2$ perturbation over an otherwise axisymmetric field, shows the presence of non-local interactions for all modes n. The analysis of such features in a realistic case may be useful to uncover some phenomena observed in physical systems.

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